theory of q-difference equations to a comparable degree of completeness. This program includes in particular the complete theory of the convergence and divergence of formal series, the explicit determination of the essential transcendental invariants (constants in the canonical form), the inverse Riemann theory both for the neighborhood of $x = \infty$ and in the complete plane (case of rational coefficients), explicit integral representation of the solutions, and finally the definition of q-sigma periodic matrices, so far defined essentially only in the case $n = 1$. Because of its extensiveness this material cannot be presented here.

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*ON THE STABILITY OF THE LINEAR FUNCTIONAL EQUATION**

BY D. H. HYERS

MATHEMATICS DEPARTMENT, UNIVERSITY OF WISCONSIN

Communicated February 18, 1941

The solutions of the linear functional equation $f(x + y) = f(x) + f(y)$ have been investigated for many spaces. Throughout this paper any solution $f(x)$ of this equation will be called a linear function or transformation. In a recent talk before the Mathematics Club of the University of Wisconsin, Dr. S. Ulam proposed the following problem of the "stability" of this equation. Suppose $f(x)$ satisfies this equation only approximately. Then does there exist a linear function which $f(x)$ approximates? To make the statement of the problem precise, let E and E' be Banach spaces and let δ be a positive number. A transformation $f(x)$ of E into E' will be called δ-linear if $\|f(x + y) - f(x) - f(y)\| < \delta$ for all x and y in E. Then the problem may be stated as follows. Does there exist for each $\epsilon > 0$ a $\delta > 0$ such that, to each δ-linear transformation $f(x)$ of E into E' there corresponds a linear transformation $l(x)$ of E into E' satisfying the inequality $\|f(x) - l(x)\| \leqq \epsilon$ for all x in E? This paper answers this question in the affirmative, and it is shown that δ may be taken equal to ϵ. This is clearly a best result, since the transformation $f(x) =$

$L(x) + c$, where $L(x)$ is linear and $\|c\| < \epsilon$, is evidently an ϵ-linear transformation for which $l(x) = L(x)$.

THEOREM 1. *Let E and E' be Banach spaces and let $f(x)$ be a δ-linear transformation of E into E'. Then the limit $l(x) = \lim_{n \to \infty} f(2^n x)/2^n$ exists for each x in E, $l(x)$ is a linear transformation, and the inequality $\|f(x) - l(x)\| \leqq \delta$ is true for all x in E. Moreover $l(x)$ is the only linear transformation satisfying this inequality.*

Proof. For any x in E the inequality $\|f(2x) - 2f(x)\| < \delta$ is obvious from the definition of δ-linear transformations. On replacing x by $x/2$ in this inequality and dividing by 2 we see that $\|(^1/_2)f(x) - f(x/2)\| < \delta/2$ for all x in E. Make the induction assumption

$$\|2^{-n}f(x) - f(2^{-n}x)\| < \delta(1 - 2^{-n}). \tag{1}$$

On the basis of the last two inequalities we find that

$$\|2^{-1}f(2^{-n}x) - f(2^{-n-1}x)\| < \delta/2$$

and

$$\|2^{-n-1}f(x) - 2^{-1}f(2^{-n}x)\| < \delta(^1/_2 - 2^{-n-1}).$$

Hence

$$\|2^{-n-1}f(x) - f(2^{-n-1}x)\| < \delta(1 - 2^{-n-1}).$$

Therefore, since the induction assumption (1) is known to be true for $n = 1$, it is true for all positive integers n and all x in E.

Put $q_n(x) = f(2^n x)/2^n$, where n is a positive integer and x is in E. Then

$$q_m(x) - q_n(x) = \frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}$$

$$= \frac{f(2^{m-n}\cdot 2^n x) - 2^{m-n}f(2^n x)}{2^m}.$$

Therefore, if $m < n$ we can apply inequality (1) and write $\|q_m(x) - q_n(x)\| < \delta(1 - 2^{m-n})/2^m$, for all x in E. Hence $\{q_n(x)\}$ is a Cauchy sequence for each x in E, and since E is complete, there exists a limit function $l(x) = \lim_{n\to\infty} q_n(x)$. Let x and y be any two points of E. Since $f(x)$ is δ-linear, $\|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)\| < \delta$ for all positive integers n. On dividing by 2^n and letting n approach infinity we see that $l(x + y) = l(x) + l(y)$. If we replace x by $2^n x$ in the inequality (1), we find that $\|f(2^n x)/2^n - f(x)\| < \delta(1 - 2^{-n})$. Hence in the limit, $\|l(x) - f(x)\| \leqq \delta$.

Suppose that there was another linear transformation $L(x)$ satisfying the inequality $\|L(x) - f(x)\| \leqq \delta$ for all x in E, and such that $l(y) \neq L(y)$ for some point y of E. For any integer $n > 2\delta/\|l(y) - L(y)\|$ it is obvious that $\|l(ny) - L(ny)\| > 2\delta$, which contradicts the inequalities

$\| L(ny) - f(ny) \| \leqq \delta$ and $\| l(ny) - f(ny) \| \leqq \delta$. Hence $l(x)$ is the unique linear solution of the inequality $\|f(x) - l(x)\| \leqq \delta$.

We have established Theorem 1 without continuity restrictions. Next we shall investigate the character of the linear transformation $l(x)$ when continuity restrictions are placed on the transformation $f(x)$.

THEOREM 2. *If under the hypotheses of theorem* 1 *we suppose that $f(x)$ is continuous at a single point y of E, then $l(x)$ is continuous everywhere in E.*

Proof. Assume, contrary to the theorem, that the linear transformation $l(x)$ is not continuous. Then there exists an integer k and a sequence of points x_n of E converging to zero such that $\| l(x_n) \| > 1/k$ for all positive integers n. Let m be an integer greater than $3k\delta$. Then $\| l(mx_n + y) - l(y) \| = \| l(mx_n) \| > 3\delta$. On the other hand,

$$\| l(mx_n + y) - l(y) \| \leqq \| l(mx_n + y) - f(mx_n + y) \| + \|f(mx_n + y) - f(y) \| + \|f(y) - l(y)\| < 3\delta$$

for sufficiently large n, since $\lim_{n\to\infty} f(mx_n + y) = f(y)$. This contradiction establishes the theorem.

COROLLARY. *If under the hypotheses of Theorem* 1 *we suppose that for each x in E the function $f(tx)$ is a continuous function of the real variable t for $-\infty < t < +\infty$, then $l(x)$ is homogeneous of degree one.*

Proof. For fixed x, $f(tx)$ is a continuous δ-linear transformation of the real axis into E'. By Theorems 1 and 2, $l(tx)$ is continuous in t and hence, being linear, $l(x)$ is homogeneous of degree one.

As Ulam has pointed out, there are many interesting generalizations of the problem to be investigated. For example, one might consider an analogous problem for transformations of a topological group which satisfy the equation for homomorphisms only approximately.

* Presented to the American Mathematical Society on April 11, 1941.